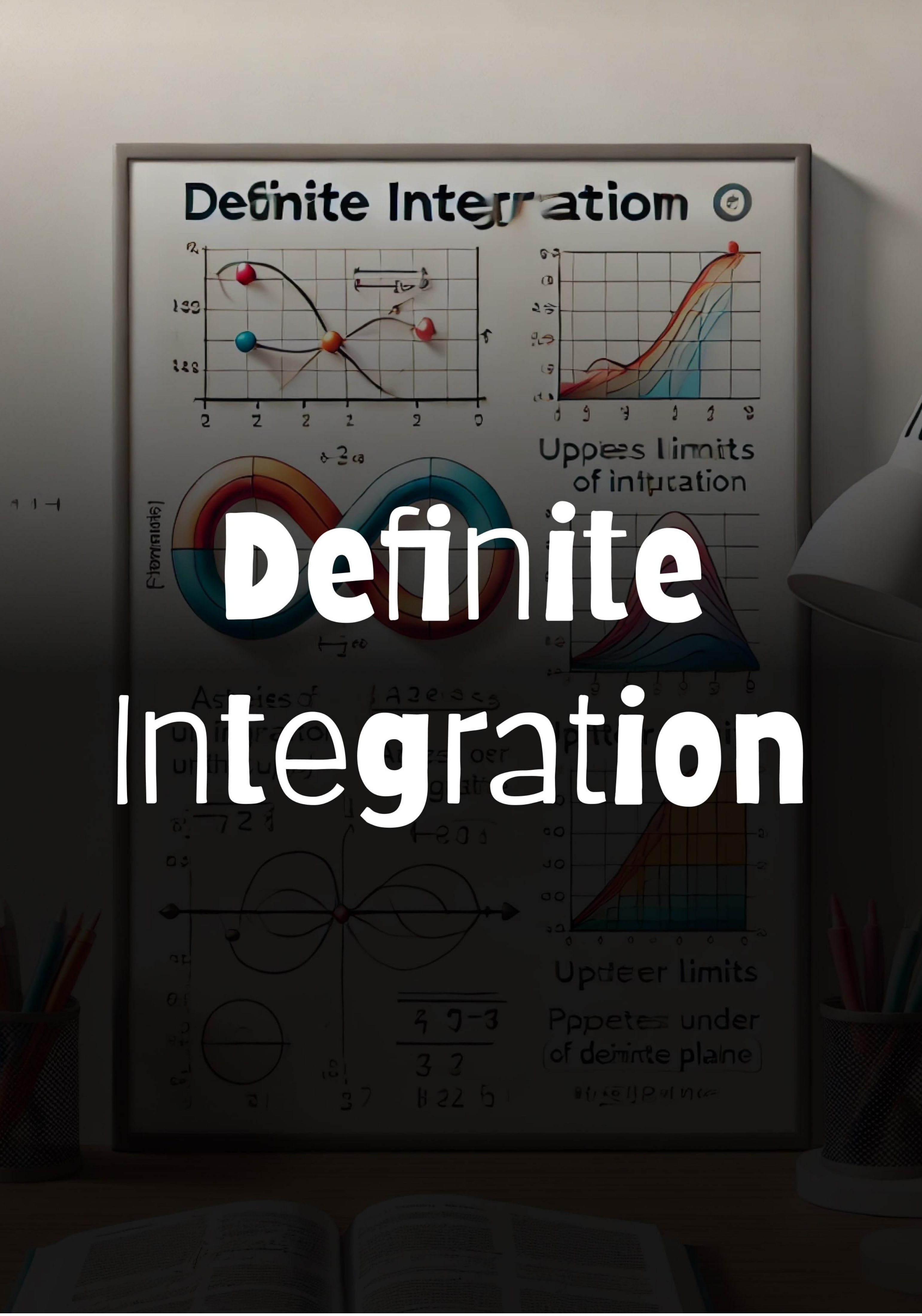
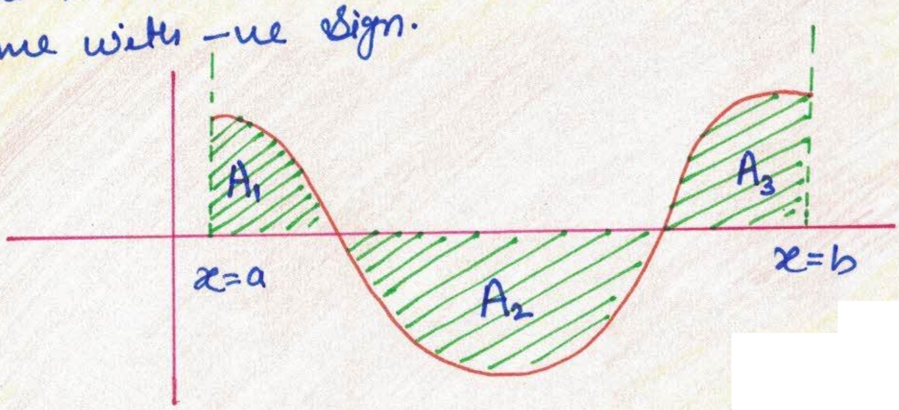
Definite Integration



DEFINITE INTEGRATION

Definition-1 $\int_a^b f(x).dx$ represent the algebrical sum of the area bounded by $y=f(x)$ with x-axis b/w the lines $x=a$ and $x=b$. Here, algebrical sum means, the area which is above x-axis will come with +ve sign and that below x-axis will come with -ve sign.



$$\int_a^b f(x).dx = A_1 - A_2 + A_3$$

$$\int_{-4}^{3} (3x-1).dx = \left[\frac{3x^2}{2} - x\right]_{-4}^{3}$$

$$\frac{3(9)}{2} - 3 - \left(\frac{3(16)}{2} + 4\right)$$

$$\frac{27}{2} - 3 - 24 - 4$$

$$= 13.5 - 31 = -17.5$$

PTR 1

If $\int_a^b f(x).dx = 0$, then it implies that the graph of $f(x)$ will cut x-axis atleast one time b/w (a,b)

But, its vice versa is not true.

Question

If $2a+3b+6c=0$, Then Prove that atleast one root of the quadratic eqn. $ax^2+bx+c=0$ lies in $(0,1)$.

$ax^2+bx+c=0$ lies in $(0,1)$

$$y\Big|_{x=0} = e \quad \left[\frac{ax^3}{3}+\frac{bx^2}{2}+cx\right]_0^1$$

$$\int_0^1 f(x)=0 \qquad \frac{a}{3}+\frac{b}{2}+c=z$$

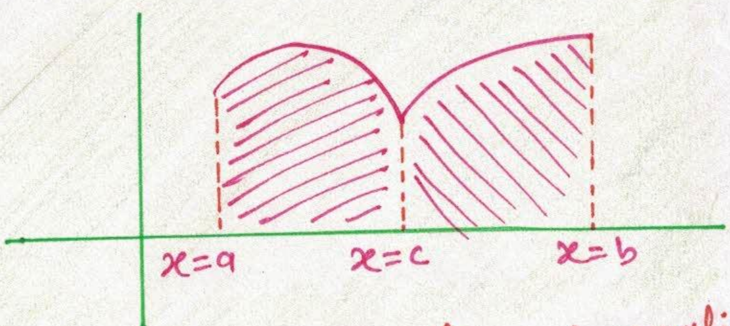
$$2a+3b+6c=6z$$

$$z=0$$

$$\text{Area}=0$$

$\Rightarrow$ one root lies b/w $(0,1)$

2. If $f(x)$ is discontinuous at $x=c$. where $c \in (a,b)$ Then $\int_a^b f(x)\cdot dx$ will be equal to $=\int_a^c f(x)\cdot dx+\int_{+c}^b f(x)dx$



3. In DI area of a line is negligible which implies $\int f(x)$ in $[a,b]$, $(a,b]$, $[a,b)$ or (a,b) is same.

4. If $f(x)>0$ in (a,b), then $\int_a^b f(x)\cdot dx>0$ Similarly,
if $f(x)<0$ in (a,b), then $\int_a^b f(x)\cdot dx<0$.

5. If $f(x)$ is above x-axis, then area is taken as +ve and if $f(x)$ is below x-axis, then it is taken as -ive.

6. $$\int_a^b f(x)\,dx \le \left|\int_a^b f(x)\cdot dx\right| \le \int_a^b |f(x)|\cdot dx.$$

Equality holds when $f(x)$ lies completely above x-axis.

7. $$\int_a^b f(x)\cdot dg(x) = \int_{g^{-1}(a)}^{g^{-1}(b)} f(x)\cdot g'(x)\cdot dx$$

Question

$$\int_{-1}^{1} x^2\cdot d(\ln x)$$

$\ln x = -1$, $x = \frac{1}{e}$ $\quad$ $\ln x = 1$, $x = e$

$$\int_{1/e}^{e} x^{\cancel{2}}\cdot\frac{1}{\cancel{x}}\cdot dx = \left[x^2/2\right]_{1/e}^{e} = \frac{e^2}{2} - \frac{1}{e^2 2}$$

$-x^2 = t^2 - 1$

Question

$$\int_0^{1/2} \frac{dx}{(1-2x)\sqrt{1-x^2}}$$

$x = 1/t$

$1 - x^2 = t^2$

$-2x\,dx = 2t\,dt$

$dx = -\frac{1}{t^2}\cdot dt$

$$\int \frac{dx}{\frac{1}{t}\cdot\frac{1}{t^2}(t^2-2)\sqrt{t^2-1}} = \int \frac{t\,dt}{(t^2-2)\sqrt{t^2-1}}$$

$t^2 - 1 = u^2$ $\quad$ $\cancel{2}t\,dt = \cancel{2}u\,du$

$$\int \frac{u\,du}{(u^2-1)u} = \int \frac{du}{u^2-1}$$

$$= \frac{1}{2}\ln\left[\frac{u-1}{u+1}\right]$$

$$= \left[\frac{1}{2}\ln\frac{\sqrt{-x^2}-1}{\sqrt{-x^2}+1}\right]_0^{1/2}$$

8

$$\int_a^b \frac{d}{dx}(f(x))\cdot dx = \int_a^b f^{2}(x)\cdot dx = \left| f(x) \right|_a^b$$

$$= f(b) - f(a)$$

provide $f(x)$ is continuous in $[a,b]$.

Question

$$\int_{-1}^{1} \frac{d}{dx}\left(\cot\left(-\frac{1}{x^2}\right)\right)\cdot dx$$

$$= \int_{-1}^{1} \frac{-1}{1+\frac{1}{x^2}}\left(-1/x^2\right)\cdot dx = \int_{-1}^{1} \frac{1}{1+x^2}\cdot dx$$

$$= \left[\tan^{-1}x\right]_{-1}^{1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \boxed{\frac{\pi}{2}}$$

Question

$$\int_{-1}^{1} \frac{d}{dx}\left(\cot^{-1}\left(\frac{1}{x}\right)\right)\cdot dx$$

$$\int_{-1}^{0^-} \frac{d}{dx}\left(\cot^{-1}\frac{1}{x}\right)\cdot dx + \int_{0^+}^{1} \frac{d}{dx}\cot^{+}(1/x)\cdot dx$$

$$= \cot^{-1} 1/x \Big|_{-1}^{0^+} + \cot^{-1}\frac{1}{x}\Big|_{0^+}^{1}$$

$$= \pi - \frac{3\pi}{4} + \frac{\pi}{4} - 0$$

$$= \pi - \frac{\pi}{2} = \boxed{\frac{\pi}{2}}$$

9

$$\lim_{x\to\infty} \int_a^b f_n(x)\cdot dx = \int_a^b \lim_{x\to\infty} f_n(x)\cdot dx$$

Question

$$\lim_{x\to\infty} \int_{-a^{1/3}}^{a^{1/3}} \left(1-\frac{t^3}{n}\right)^n \cdot t^2 \cdot dt = \frac{2\sqrt{2}}{3} \quad \text{find } a.$$

$$\lim_{x\to\infty} \left(1-\frac{t^3}{n}\right)^n \cdot 0 = e^{-t^3}$$

$$I = \int_{-a^{1/3}}^{a^{1/3}} e^{-t^3} t^2 dt$$

$$-t^3 = y$$
$$-3t^2 dt = dy$$

$$I = \int_{-a^{1/3}}^{a^{1/3}} e^y \cdot \frac{dy}{3} \qquad I = -\frac{1}{3} \cdot e^y \Big|_{-a^{1/3}}^{a^{1/3}}$$

$$I = -\frac{1}{3}\left[e^a - e^{-a}\right]$$

$$-\frac{1}{\cancel{3}}\left[e^a - e^{-a}\right] = \frac{2\sqrt{2}}{\cancel{3}}$$

$$-\left(\frac{e^{2a}-1}{e^a}\right) = 2\sqrt{2}$$

$$z^2 - 1 = 2\sqrt{2}\, z$$

$$z^2 + 2\sqrt{2}\, z - 1 = 0$$

$$z = \frac{-2\sqrt{2} \pm \sqrt{8+4}}{2} \Rightarrow \boxed{z = -\sqrt{2} \pm \sqrt{3}}$$

10 if $g(x)$ is inverse of $f(x)$ and domain of $f(x)$ is $[a,b]$ such that $f(a)=c$ and $f(b)=d$, then

$$\int_a^b f(x)\cdot dx + \int_c^d g(x)\cdot dx = bd - ac$$

As $g(x)$ is inverse of $f(x) \Rightarrow g(f(x)) = x$

$$f(x)\cdot x \Big|_a^b - \int_a^b x \cdot f'(x)\cdot dx + \int_c^d g(x)\cdot dx$$

$$\int_a^b g(f(x)) \cdot f'(x) \cdot dx$$

$$f(x) = t$$

$$f'(x) \cdot dx = dt$$

$$\int_c^d g(t) \cdot dt$$

$$x \cdot f(x)\Big|_a^b - \int_c^d g(t) \cdot dt + \int_c^d g(x)\,dx$$

$$= f(b) \cdot b - f(a) \cdot a$$

$$= bd - ac$$

Question

$$\int_0^1 e^{\sqrt{e^x}} \cdot dx + \int_e^{e^{\sqrt{e}}} 2\ln(\ln x) \cdot dx$$

$$e^{\sqrt{e}}$$

$$\int_0^{\pi/2} \sin x \cdot dx \quad \text{or} \quad \int_0^{\pi/2} \cos x \cdot dx = 1$$

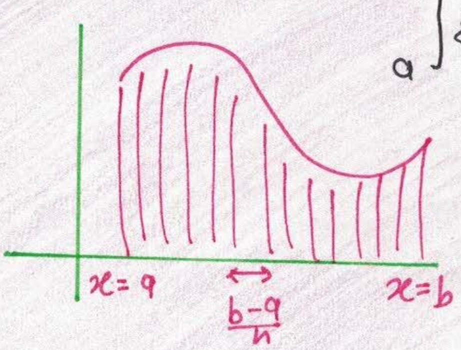
$$\int_0^{\pi/4} \sin^2 x \cdot dx = \int_0^{\pi/2} \cos^2 x \cdot dx = \pi/4$$

$$\int_0^{\pi/2} \sin^3 x\,dx = \int_0^{\pi/2} \cos^3 x\,dx = 2/3$$

$$\int_0^{\pi/2} \sin^4 x\,dx = \int_0^{\pi/2} \cos^4 x\,dx = 3\pi/16$$

DEFINITION 2 ...

$$\int_a^b f(x) \cdot dx = \lim_{x \to \infty} \sum_{r=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + \frac{(b-a)r}{n}\right)$$



$$a + \frac{b-a}{n}$$

Area of the strip

$$\lim_{x \to \infty} \left[\left(\frac{b-a}{n}\right) f\left(a\left(\frac{b-a}{n}\right) r\right)\right]$$

$$\Rightarrow \int_a^b f(x) \cdot dx = \lim_{x \to 0} \sum_{r=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + \frac{(b-a)r}{n}\right)$$

Question

$$\int_{-4}^{3} (3x-1)\,dx :\longrightarrow \lim_{x\to\infty} \sum_{r=1}^{n} \frac{7}{n} f\left[-4+\frac{7r}{n}\right]$$

$$\Rightarrow \lim_{x\to\infty} \sum_{r=1}^{n} \frac{7}{n}\left[3\left(-4+\frac{7r}{n}\right)-1\right]$$

$$\Rightarrow \lim_{x\to\infty} \sum_{r=1}^{n} \frac{7}{n}\left(-13+\frac{21r}{n}\right)$$

$$= \sum_{r=1}^{n}\left(-\frac{91}{n}\right) + \sum \frac{147r}{n^2}$$

$$= \lim_{x\to\infty}\left[\frac{-91}{n}\times n + \frac{147}{n^2}\left(\frac{n^2+n}{2}\right)\right]$$

$$= -91+\frac{147}{2} \quad \boxed{= -\frac{35}{2}}$$

DEFINITION 3

$$\int f(x)\cdot dx = g(x)+C = \int_a^b f(x)\cdot dx = g(b)-g(a)$$

Variable of $\int$ in D.I is dummy variable.

$$\int_a^b f(x) = L = \int_a^b (t)\,dt = \int_a^b f(y)\cdot dy = \int_a^b f(K)\cdot dK$$

$$\int_a^b f(x)\cdot dx = -\int_b^a f(x)\,dx$$

if we use substitution rule in D.I for x Then it is must that limit of $\int$ will b change by substitution rule and Dont bring back the old variable.

NOTE

The substitution of variable will be in such a way that it has ~~do~~ to be continuous within the limit of integration.

$$\int_a^b f(x)\cdot dx = \int_{-b}^{-a} f(-x)\cdot dx$$

$-x = t \Rightarrow -dx = dt \quad \int_b^a f(t)\,dt = \int_a^b f(x)\cdot dx$

$$\int_a^b f(x)\cdot dx = \int_{a-c}^{b-c} f(x+c)\cdot dx$$

Question $\int_{26}^{5} \frac{2x}{1+x^2}\cdot dx \Rightarrow 1+x^2 = t \qquad 2x\cdot dx = dt$

$$\int_5^{26} \frac{dt}{t} = \boxed{\ln\frac{26}{5}}$$

$$\int_a^b f(x)\,dx + \int_a^b g(x)\cdot dx + \int_a^b h(x)\cdot dx = \int_a^b (fx + gx + hx)\,dx$$

Here we have to keep in mind that limit of $\int$ in each D.I must be same.

If $f(x)$ is piece cosx continuous f^n $[a,b]$ where $c_1, c_2, c_3 \text{-----} c_n \in [a,b]$ are point of discontinuity.

$$\int_a^b f(x)\,dx = \int_a^{c_1} fx\,dx + \int_{c_1}^{c_2} f(x)\cdot dx \text{------} \int_{c_n}^{b} f(x)\cdot dx$$

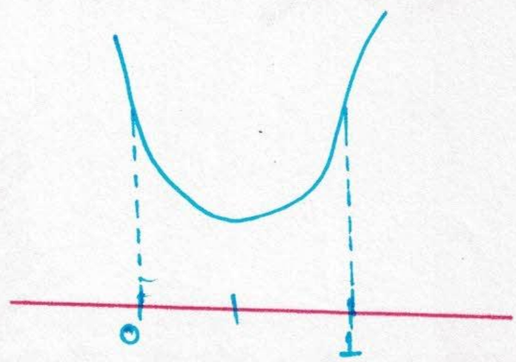
Question

$$\int_0^2 [x^2 - x + 1]\,dx$$

$x^2 - x + 1 = 1$ $\qquad$ $x^2 - x + 1 = 2$

$x = 0 \quad x = 1$ $\qquad$ $x = \dfrac{1+\sqrt{5}}{2}$ $\qquad$ $x = \dfrac{1-\sqrt{5}}{2}$

$$\int_0^1 0\cdot dx + \int_1^{\frac{1+\sqrt{5}}{2}} 1\cdot dx + \int_{\frac{1+\sqrt{5}}{2}}^{2} 2\cdot dx$$



$$\frac{1+\sqrt{5}}{2} - 1 + 2\left(2 - \left(1 - \frac{\sqrt{5}}{2}\right)\right)$$

$$3 - \left(\frac{1+\sqrt{5}}{2}\right)$$

$$\frac{6 - (1+\sqrt{5})}{2} = \boxed{\frac{5-\sqrt{5}}{2}}$$

Question

$$\int_{-1}^{1} [x + [x + [x + [x]]]]\,dx$$

$$[x+k] = [x] + k = \qquad k \in I$$

$$\int_{-1}^{1} [x + [x + [x] + [x]]\,dx$$

$$\Rightarrow \int_{-1}^{1} [x + [x] + [x] + [x]]\,dx$$

$$\Rightarrow \int_{-1}^{1} [x] + [x] + [x] + [x]\,dx$$

$$\Rightarrow 4\int_{-1}^{1} [x]\,dx \Rightarrow 4\int_{-1}^{0} -1\,dx + \int_0^1 0\,dx$$

$$\boxed{\Rightarrow -4}$$

Question

$$\int_0^{5050\pi} [\tan^{-1}x \cdot] \cdot dx$$

$$\int_0^{\tan 1} 0\cdot dx + \int_{\tan 1}^{5050\pi} 1\cdot dx \Rightarrow 5050\pi - \tan 1$$

$\{ 0 - \frac{\pi}{2}$ we $[\tan^{-1}x]$ discont. $\quad x = \tan 1$

$$\int_1^{10\pi} [\sec^{-1}x] + [\cot^{-1}x] \cdot dx$$

$\pi = 3.14$

$[0, \pi]$

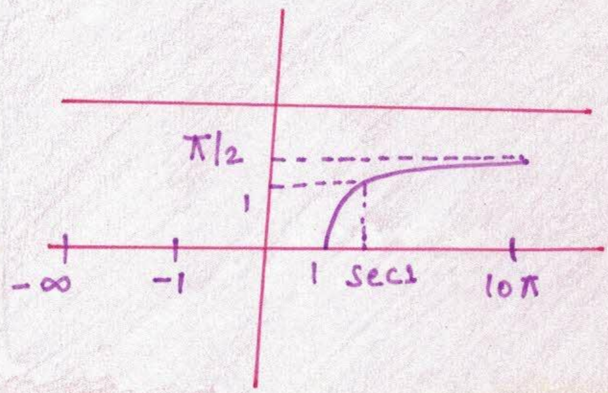
$1, 2, 3$

$$\int_1^{10\pi} [\sec^{-1}x] \cdot dx + \int_1^{10\pi} [\cot^{-1}x]\, dx$$

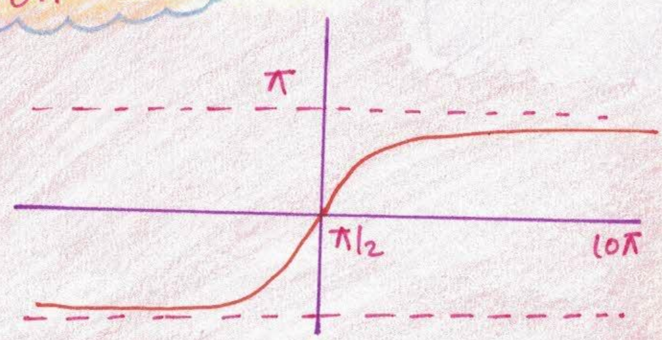
$[0, \pi]$

$2, 3$

$$\int_1^{\sec 1} 0\cdot dx + \int_{\sec 1}^{10\pi} 1\cdot dx + \int_1^{10\pi} 0\cdot dx$$



$\Rightarrow 10\pi - \sec 1$



★★★ KING'S FORMULA / GOD FORMULA ...

$$\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$$

PROOF

$$a+b-x=t$$

$$-dx=dt$$

$$\int_b^a -f(t)\cdot dt = \int_a^b f(t)\cdot d \Rightarrow \int_a^b f(x)\cdot dx$$

$$I=\int_0^{\pi/2}\frac{\sin^3 x\,dx}{\sin x+\cos x} \quad \text{use king} = \int_0^{\pi/2}\frac{\sin^3(\pi/2-x)\cdot dx}{\sin(\pi/2-x)+\cos(\pi/2-x)}$$

$$I=\int_0^{\pi/2}\frac{\cos^3 x}{\cos x+\sin x}\,dx$$

$$\text{king + add} = \int_0^{\pi/2}\frac{\sin^3 x+\cos^3 x}{(\sin x+\cos x)}\cdot dx$$

$$=\int_0^{\pi/2}\frac{\cancel{\sin x+\cos x}\left(1-\frac{2\sin x\cos x}{2}\right)\cdot dx}{\cancel{\sin x+\cos x}}$$

$$=\left[x+\frac{\cos 2x}{4}\right]_0^{\pi/2} = \frac{\pi}{2}-\frac{1}{4}+\frac{1}{4} = \boxed{\frac{\pi}{2}}$$

$$\int_{\pi/6}^{\pi/3}\sin 2x\,\ln(\tan x)\cdot dx = \int_{\pi/6}^{\pi/3}\sin 2\left(\frac{\pi}{2}-2x\right)\ln\left(\tan\left(\frac{\pi}{2}-2x\right)\right)dx$$

$$\frac{\pi}{6}+\frac{\pi}{3}=\frac{\pi+2\pi}{6}=\frac{\pi}{2}$$

$$=\int_{\pi/6}^{\pi/3}\sin 2x\cdot\ln(\cot x)\cdot dx$$

$$2I=\int_{\pi/6}^{\pi/2}\sin 2x\,(\ln 1) \Rightarrow \boxed{I=0}$$

$$\int_{50}^{100}\frac{\ln x}{\ln(x)+\ln(150-x)}$$

$$= \int_{50}^{100} \frac{\ln(150-x)}{\ln(150-x)} \Rightarrow \int 1 - \frac{\ln(150-x)\cdot dx}{\ln x + \ln(150-x)}$$

$$I = \int_{50}^{100} \frac{\ln(150-x)}{\ln(150-x) + \ln x}$$

$$2I = \int_{50}^{100} \frac{\ln x + \ln(150-x)}{\ln(150-x) + \ln x}$$

$$2I = \int_{50}^{100} \frac{\ln((150-x)\cdot x)}{\ln(150-x)\cdot x)} \cdot dx$$

$$2I = 50$$

$$\boxed{I = 25}$$

$$\int_0^{\pi} \frac{dx}{1+2^{\tan x}} \Rightarrow I = \int_0^{\pi} \frac{dx}{1+2^{-\tan x}}$$

$$2I = \int_0^{\pi} \frac{2^{\tan x}+1}{2^{\tan x}+1} \cdot dx$$

$$2I = \pi$$

$$\boxed{I = \pi/2}$$

$$\int_0^{\pi/2} \frac{x \sin x \cdot \cos x}{\sin^4 x + \cos^4 x}$$

$$2(\pi/2 - x)$$
$$\Rightarrow \pi - 2x$$

$$I = \frac{1}{2} \int_0^{\pi/2} \frac{x \sin 2x}{\sin^4 x + \cos^4 x} \cdot d$$

$$(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x$$

$$1 - \frac{\sin^2 2x}{2}$$

$$\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} \cdot dx$$

$$I \Rightarrow \int \frac{(\pi/2 - x) \sin x \cos x}{\sin^4 x + \cos^4 x}$$

$$2I = \int_0^{\pi/2} \frac{\sin x \cos x \left[x + \pi/2 - x\right]}{\sin^4 x + \cos^4 x}$$

$$2I = \frac{\pi}{4} \int_0^{\pi/2} \frac{\sin 2x}{1 - \frac{1}{2} \sin^2 2x} \cdot dx$$

$$2I = \frac{\pi}{2} \int \frac{\tan x \sec^2 x}{\tan^4 x + 1} \Rightarrow \frac{\pi}{2} \int_0^{\pi/2} \frac{dt}{2(t^2+1)}$$

$$I = \int_0^{\pi/2} \frac{\pi}{8} \tan^{-1} \tan^2 x$$

$$\int_0^1 \cot^{-1}(1 - x + x^2)\, dx \Rightarrow I = \int \cot^{-1}\left(1 - (1-x) + (1-x)^2\right)$$

$$I = \int_0^1 \cot^{-1}(x + x^2 - x + 1) \cdot dx$$

$$2I = \int \cot^{-1}(1 - x + x^2)\, dx + \pi/2 - \tan^{-1}(1 - x + x^2)\, dx$$

$$\int_0^1 \tan^{-1}\left(\frac{1}{x^2 - x + 1}\right) \cdot dx \Rightarrow \int_0^1 \tan^{-1}\left(\frac{x - (x-1)}{1 + x(x-1)}\right) \cdot dx$$

$$\Rightarrow \int_0 \tan^{-1}(x) - \tan^{-1}(x-1) \cdot dx$$

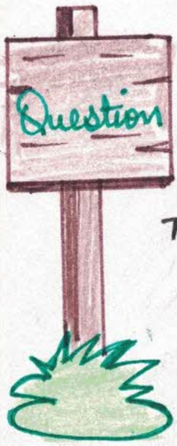
$$\int_0^1 \tan^{-1}(x) - \int_0 \tan^{-1}(\cancel{1} - x + \cancel{1}) \cdot dx$$

$$I = \int_0^1 \tan^{-1}x + \int_0^1 \tan^{-1}x \cdot dx$$

$$I = 2\int_0^1 \tan^{-1}x\,dx = 2\left[\tan^{-1}x \cdot x - \frac{1}{2}\int \frac{2x}{x^2+1}\cdot dx\right]_0^1$$

$$I = 2\left[x\tan^{-1}x - \frac{1}{2}\ln(x^2+1)\right]_0^1$$

$$I = 2\left[\pi/4 - \frac{\ln 2}{2}\right] \Rightarrow \boxed{I = \pi/2 - \ln 2}$$



$$\int_0^{\pi/4} \frac{x}{1+\cos 2x+\sin 2x}\cdot dx$$

$$\int_0^{\pi/4}\frac{(\pi/4 - x)}{1+\cos(\pi/2-2x)+\sin(\pi/2-2x)} = \int_0^{\pi/4}\frac{\pi/4 - x}{1+\sin 2x+\cos 2x}$$

$\tan x = t$

$$I = \frac{\pi}{8}\int \frac{1+t^2\,(dx}{1+\cancel{t^2}+2t+1-\cancel{t^2}} \qquad 2I = \int\frac{\pi/4}{1+\sin 2x+\cos 2x}\cdot dx$$

$$\boxed{= \frac{\pi}{8}\int\frac{dt}{t+1} = \left[\frac{\pi}{16}\ln(\tan x)\right]_0^{\pi/4}}$$

If $f(x)$ is an even f^n, then evaluate -

$$\int_{-2}^{2}\left(x^3 f(x) + x f''(x) + 2\right)\cdot dx$$

$$\int_{-2}^{2} \underset{\mathrm{I}}{x^3}\, \underset{\mathrm{II}}{f(x)} \cdot dx + \int_{-2}^{2} x f''(x) \cdot dx + \left[2x\right]_{-2}^{2}$$

$$I = \int_{-2}^{2} \left(-x^3 f(x) - x f''(x) + 2\right) \cdot dx$$

$$2I = \left[4x\right]_{-2}^{2} \qquad \boxed{I = 2[4] = 8}$$

NOTE

$$\boxed{\int_0^\infty \frac{\ln x \cdot dx}{ax^2 + bx + a} = 0}$$

Here, coeff of x^2 and constant must be same.

Question

$$\int_0^\infty \frac{\ln x}{x^2 + 2x + 4} \cdot dx \qquad \begin{aligned} x &= 2t \\ dx &= 2dt \end{aligned}$$

$$\int \frac{\ln(2t)\, 2 \cdot dt}{4t^2 + 4t + 4} = \frac{2}{4} \int \frac{\ln(2t)\, dt}{t^2 + t + 1}$$

$$= \frac{1}{2} \int \frac{\ln 2\, dt}{t^2 + t + 1} + \frac{1}{2} \int \frac{\ln t\, dt}{t^2 + t^2 + 1}$$

$$= \frac{1}{2} \ln \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{\ln 2}{2} \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}t}{2}\right)\Big]_0^\infty$$

$$\boxed{= \frac{\ln 2}{\sqrt{3}} \cdot \frac{\pi}{2}}$$

PROPERTY 7

$$\int_a^b x\cdot f(x)\cdot dx = \frac{a+b}{2}\int_a^b f(x)\cdot dx$$

Provided $f(a+b-x) = f(x)$

$I \int_a^b x f(x)\cdot dx$

King $I = \int_a^b (a+b-x)\cdot f(a+b-x)\cdot dx$

$2I = \int_a^b x f(x) + \left((a+b)-x\right) f(x)\, dx$

$I = \frac{a+b}{2}\int_a^b f(x)\cdot dx$

$I \quad a+b \int_a^b f(x)\,dx$

If $I_1 = \int_0^\pi x\cdot f(\sin^3 x + \cos^2 x\cdot dx$

$I_2 \int_0^\pi f(\sin^3 x \cos^2 x\cdot dx$

Then find the value of $\frac{I_1}{I_2} = ?$

$I_1 = \frac{\pi}{2}\int f(\sin^3 x + \cos^2 x)\cdot dx$

PROPERTY 8

$$\int_{-a}^{a} f(x)\cdot dx = \int_0^a f(x)\cdot dx + \int_0^a f(-x)\cdot dx$$

$$\int_a^0 f(x)\cdot dx = \int_{-a}^0 f(x)\cdot dx + \int_0^a f(x)\cdot dx$$

$x = -t \qquad dx = -dt$

$$I = \int_a^0 -f(-t)\,dt + \int_0^a f(x')\,dx$$

$$\Rightarrow \int_0^a f(-x)\cdot dx + \int_0^a f(x)\cdot dx$$

This property is useful only when limit of Definite Integration are symmetrical.

PROPERTY-9

$$\int_{-a}^{a} f(x)\cdot dx = \begin{cases} 0 & \text{when } f(x) \text{ is odd} \\ 2\int_0^a f(x)\cdot dx & \text{when } f(x) \text{ is even} \end{cases}$$

$$\int_{-3}^{3} \frac{x^2 \sin x}{x^4 x^2 + \cos x}\cdot dx = 0$$

{ Limit symmetrical, odd fn }

QUESTIONS ON PROPERTY

Question $\int_0^4 [x]\cdot dx = ?$

$$\int_0^1 0\cdot dx + \int_1^2 1\cdot dx + \int_2^3 2\cdot dx + \int_3^4 3\cdot dx$$

$$\boxed{= (2-1) + 2(3-2) + 3(4-3) = 6}$$

Question $\int_0^2 [x^2]\, dx = ?$

$$\int_0^1 0\cdot dx + \int_1^{\sqrt{2}} 1\cdot dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\cdot dx + \int_{\sqrt{3}}^{2} 3\cdot dx$$

$$= \sqrt{2}-1 + (\sqrt{3}-\sqrt{2})\cdot 2 + 3(-\sqrt{3}+2)$$

$$\boxed{= 5-\sqrt{2}-\sqrt{3}}$$

Question $\int_0^{3/2} x[x^2]\cdot dx = ?$

$$\int_0^1 0\cdot x\,dx + \int_1^{\sqrt{2}} x\cdot dx + \int_{\sqrt{2}}^{3/2} 2x\cdot dx$$

$$= \frac{x^2}{2}\Big|_1^{\sqrt{2}} + \frac{2x^2}{2}\Big|_{\sqrt{2}}^{3/2} = \boxed{\frac{1}{2}+\frac{1}{4} = \frac{3}{4}}$$

Question $\int_0^3 |5x-9|\cdot dx$?

$$\int_0^{9/5} (9-5x)\cdot dx + \int_{9/5}^{3} (5x-9)\cdot dx$$

PROPERTY - 10

QUEEN'S FORMULA

$$\int_0^a f(x)\cdot dx = \begin{cases} 0 & f(a-x) = -f(x) \\ 2\int_0^{a/2} f(x)\cdot dx & f(a-x) = f(x) \end{cases}$$

$$I = \int_0^a f(a-x)\cdot dx$$

CASE-I

$$f(a-x) = -f(x)$$

$$2I = \int_0^a f(x) + f(a-x) \cdot dx$$

$$2I \int_0^a f(x) - f(x)\, dx \qquad \boxed{I = 0}$$

CASE-II

$$f(a-x) = f(x)$$

$$I = \int_0^a f(x) \cdot dx$$

$$I = \int_0^{a/2} f(x) \cdot dx + \int_{a/2}^{a} f(x) \cdot dx$$

$x = a - t$
$dx = -dt$

$$I = \int_0^{a/2} f(x) \cdot dx + \int_{a/2}^{0} -f(a-t) \cdot dt$$

$$I = \int_0^{a/2} f(x) \cdot dx + \int_0^{a/2} f(a-x) \cdot dx$$

$$I = \int_0^{a/2} f(x) \cdot dx + \int_0^{a/2} f(x) \cdot dx = \boxed{2\int_0^{a/2} f(x) \cdot dx}$$

Working Rule of Definite Integration if the integrand is in the form of $\int_0^a f(x) \cdot dx$ Then Check.

$f(a-x) = -f(x) \rightarrow$ if yes $\rightarrow = 0$

$\downarrow$

$f(a-x) = f(x)$ if yes $\rightarrow 2\int_0^{a/2} f(x)\cdot dx$

$\downarrow$ NO

Check whether $\int_0^a f(x)\cdot dx$ is easy to solve or $\int_0^a f(a-x)\cdot dx$ is easy.

$\downarrow$ Otherwise

Use King + Add

Question $\int_0^{2\pi} |\sin x|\cdot dx$

$$f(2\pi - x) = |\sin(2\pi - x)| = |\sin x| = f(x)$$

$$I = 2\int_0^{\pi} |\sin x|\cdot dx$$

$$f(\pi - x) = |\sin(\pi - x)| = |\sin x| = f(x)$$

$$I = 4\int_0^{\pi/2} |\sin x|\cdot dx$$

$$f(\pi/2 - x) = |\sin(\pi/2 - x)| = |\cos x|$$

$$I = 4\int_0^{\pi/2} \sin x\cdot dx = -4\left[\cos x\right]_0^{\pi/2} \quad \boxed{= 4}$$

Question

$$\int_0^{2\pi} \cos^5 x \, dx = 0$$

$$2\int_0^{2\pi} \cos^5 (2\pi - x) \cdot dx$$

$$\boxed{= 2\int_0^{\pi} \cos^5 x \, dx = 0}$$

Question

$$\int_0^{2\pi} \frac{x \cdot \sin^4 x}{\sin^4 x + \cos^4 x} \cdot dx$$

$$2I = \int_0^{2\pi} \frac{x \sin^4 x + (2\pi - x) \sin^4 x}{\sin^4 x + \cos^4 x} \cdot dx$$

$$2I = \int_0^{2\pi} \frac{2\pi \sin^4 x}{\cos^4 x + \sin^4 x} \cdot dx$$

$$I = \pi \int_0^{\pi} \frac{\sin^4 x}{\cos^4 x + \sin^4 x} \cdot d$$

$$I = 2\pi \int_0^{\pi} \frac{\sin^4 x}{\cos^4 x + \sin^4 x} \cdot dx$$

$$I = 4\pi \int_0^{\pi/2} \frac{\sin^4 x}{\cos^4 x + \sin^4 x} \cdot dx$$

King + Add

$$2I = 4\pi \int_0^{\pi/2} \frac{\sin^4 x + \cos^4 x}{\cos^4 x + \sin^4 x} \cdot d$$

$$I = 2\pi \int_0^{\pi/2} dx = 2\pi \left(\pi/2\right)$$

$$\boxed{I = \pi^2}$$

Question

$$\int_0^{\pi/2} \ln(\sin x)\cdot dx$$

King + Add

$$2I = \int_0^{\pi/2} (\ln \sin 2x - \ln 2)\cdot dx$$

$$2I = \int_0^{\pi/2} \ln(\sin 2x)\cdot dx - \int_0^{\pi/2} \ln 2 \cdot dx$$

$$2I = \frac{2}{2}\int_0^{\pi/2} \ln \sin t \;dt - \pi/2 \ln 2$$

$$2I = I - \pi/2 \ln 2$$

$$\boxed{I = -\pi/2 \ln 2}$$

NOTE

- $$\int_0^{\pi/2} \ln(\sin x)\cdot dx = \int_0^{\pi/2} \ln(\cos x)\cdot dx = -\frac{\pi}{2}\ln 2$$
- $$\int_0^{\pi/2} \ln(\tan x)\cdot dx = \int_0^{\pi/2} \ln(\cot x)\cdot dx = 0$$
- $$\int_0^{\pi/2} \ln(\sec x)\cdot dx = \int_0^{\pi/2} \ln(\operatorname{cosec} x)\cdot dx = \frac{\pi}{2}\ln 2$$

$$\int_{-1}^{1} \frac{dx}{(1+x^2)(1+e^x)}$$

$$= \int_0^1 \frac{dx}{(1+x^2)(1+e^x)} + \int_0^1 \frac{e^x\cdot dx}{(1+x^2)(1+e^x)}$$

$$= \int_0^1 \frac{e^x+1}{(1+x^2)(1+e^x)} \cdot dx$$

$$= \int_0^1 \frac{1}{1+x^2} \cdot dx = \tan^{-1} x = \frac{\pi}{4}$$

$$\int_0^1 \frac{\sin^{-1} x}{x} \cdot dx$$

$x = \sin\theta \qquad \theta \in (0, \pi/2)$

$$\int_0^{\pi/2} \frac{\theta}{\sin\theta} \cos\theta \, d\theta = \int_0^{\pi/2} \theta \cot\theta \cdot d\theta$$

$$= \int_0^{\pi/2} \cot\theta \cdot d\theta - \int_0^{\pi/2} 1 \cdot \int \underbrace{\cot\theta}_{\ln \sin\theta} \cdot d\theta$$

$$= \left[\theta \ln |\sin\theta|\right]_0^{\pi/2} + \frac{\pi}{2} \ln 2$$

$$\boxed{I = \frac{\pi}{2} \ln 2}$$

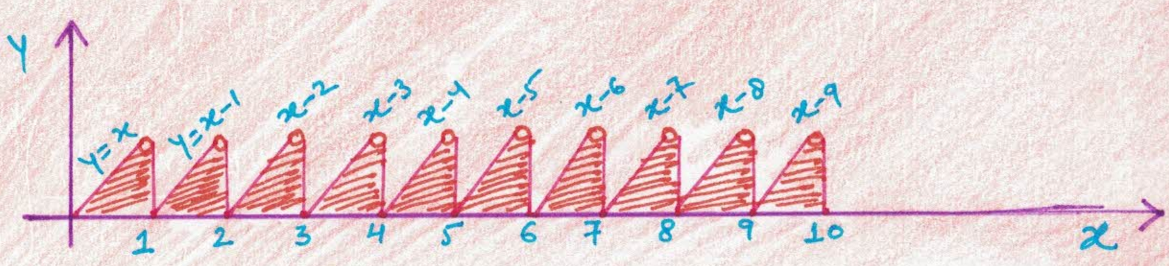
JACK FORMULA

$\int_0^{nT} f(x) \cdot dx = n \int_0^{T} f(x) \cdot dx$, where $f(x)$ is a periodic f^n $x \in$ Integer with fundamental Period 'T'.

Question

$$\int_0^{10}\{x\}\cdot dx$$

$$\int_0^{10.1}\{x\}\cdot dx = 10\int_0^1\{x\}\cdot dx$$



$$= 10\int_0^1 x_0\, dx = 10\left[\frac{x^2}{2}\right]_0^1 = \frac{\cancel{10}^{5}}{\cancel{2}}[1] = \boxed{5}$$

Question

$$\int_0^{1000} e^{x-[x]}\cdot dx$$

$$1000\int_0^1 e^{\{x\}}\cdot dx = 1000\int_0^1 e^{x}\cdot dx$$

$$= 1000\,(e^1 - e^0)$$

$$\boxed{= 1000\,(e-1)}$$

Question

$$\int_0^{200\pi}\frac{dx}{1+e^{\sin x}}$$

$$I = 100\int_0^{2\pi}\frac{dx}{e^{\sin x}+1}$$

$$2I = 100\int_0^{2\pi}\frac{\cancel{e^{\sin x}+1}\cdot dx}{\cancel{e^{\sin x}+1}}$$

$$2I = \frac{100\,(2\pi)}{1} = \boxed{I = 100\pi}$$

PROPERTY # 12

$$\int_a^{a+nT} f(x)\cdot dx = n\int_0^T f(x)\cdot dx$$

PROPERTY # 13

$$\int_{mT}^{nT} f(x)\cdot dx = (n-m)\int_0^T f(x)\cdot dx$$

PROPERTY # 14

$$\int_T^{T+a} f(x)\cdot dx = \int_0^a f(x)\cdot dx$$

Question

$$\int_0^{n\pi+v} |\cos x|\cdot dx = ?$$

$\frac{\pi}{2} < v < \pi$

$x \in N$

$$\int_0^{n\pi} |\cos x|\, dx + \int_{n\pi}^{n\pi+v} |\cos x|\, dx$$

$$= n\int_0^{\pi} |\cos x|\, dx + \int_0^{v} |\cos x|\cdot dx$$

$$= 2n\int_0^{\pi/2} \cos x\, dx + \int_0^{\pi/2} \cos x\cdot dx + \int_{\pi/2}^{v} (-\cos x)\cdot dx$$

$$= 2n + 1 - \left[\sin x\right]_{\pi/2}^{v} \Rightarrow \boxed{2n + 2 - \sin v}$$

Question

$$\int_0^{\pi} x\left(\sin^2\cdot\sin x + \cos^2\cos x\right)\cdot dx$$

$$2I = \pi\int_0^{\pi} \left(\sin^2\sin x + \cos^2\cos x\right)\cdot dx$$

$$2I = 2\pi\int_0^{\pi/2} \sin^2\sin x + \cos^2\cos x\cdot dx$$

$$I = \pi \int_0^{\pi/2} \sin^2 \cos x + \cos^2 \sin x \cdot dx$$

$$\boxed{2I = \pi \int_0^{\pi/2} 2 \cdot dx}$$

Question

If $f(x) + f(x+4) = f(x+2) + f(x+6)$

If $\int_x^{x+8} f(x) \cdot dx = g(x)$, then $g(4) = ?$

$x = x+2$

$$f(x+2) + f(x+6) = f(x+4) + f(x+8)$$

$$f(x+2) + f(x+6) = f(x) + f(x+4)$$

$$f(x+8) = f(x)$$

Periodic with Period $T = 8$

$$\boxed{\int_0^8 f(x) \cdot dx = g(x) = \text{Const.}}$$

NOTE

$\int_0^T f(x) \cdot dx = 0$ if $f(x)$ is an odd function.

NEWTON'S LEIBNITZ THEOREM

FORM I

$$y = \int_{f_1(x)}^{f_2(x)} g(x) \cdot dx$$

$$dy/dx = g(f_2(x)) \cdot f_2'(x) - g(f_1(x)) \cdot f_1'(x)$$

Question

$$f(x) = \int_{3}^{e^{3x}} \frac{t \cdot dt}{\ln t}$$

$$f'(x) = \frac{3e^{3x} \cdot e^{3x}}{\ln e^{3x}} - \frac{3}{\ln 3} \cdot 0$$

$$\boxed{= \frac{3e^{3x}}{3x}}$$

Question

$$x = \int_{0}^{y} \frac{dt}{\sqrt{1+4t^2}}$$

$$\frac{dx}{dy} = \frac{1}{\sqrt{1+4y^2}} \cdot 1$$

$$\frac{dy}{dx} = \sqrt{1+4y^2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2\sqrt{1+4y^2}} \cdot 8y \cdot \frac{dy}{dx} = \frac{1}{2\sqrt{1+4y^2}} \cdot 8y \cdot \sqrt{1+y^2}$$

$$\boxed{\frac{d^2y}{dx^2} = 4y \qquad K = 4}$$

Question

$$\int_{0}^{2n\pi} \max\{\sin x, \sin^{-1} \sin x\} \cdot dx$$

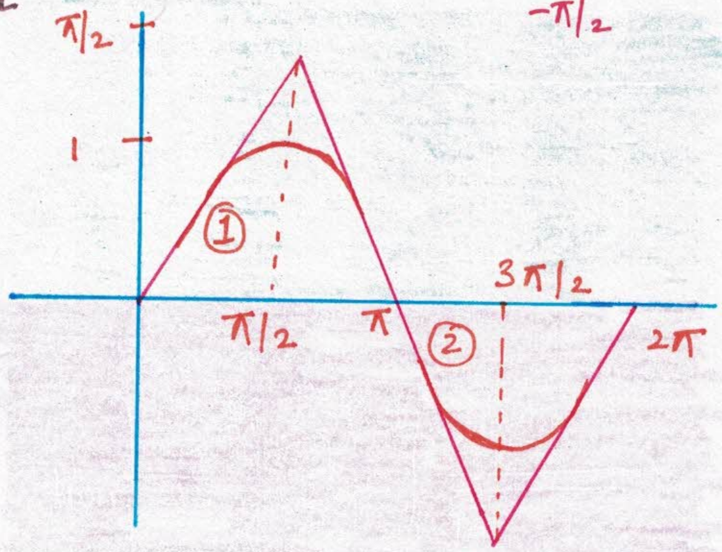
Question

Let $f(x)$ is continuous $f(x) > 0 \;\forall\; x \geq 0$ if

$$[f(x)]^{101} = 1 + \int_{0}^{x} f(t) \cdot dt \cdot \quad f(101)^{100} = ?$$

Question Find max. value of f

$$f(x) = \sin x + \int_{-\pi/2}^{\pi/2} (\sin x + t\, f(x)) \cdot dt$$



area ① $= \frac{1}{2} \times \pi \times \pi/2$

$= \frac{\pi^2}{4}$

area ② $= \int_0^{\pi} \sin x \cdot dx$

$= [\cos x]_0^{\pi}$

$= -1 - 1 = -2$

$$\boxed{1 - n\left[\frac{\pi^2}{4} - 2\right]}$$

Question

$$f(x)^{101} = 1 + \int_0^x f(t) \cdot dt$$

$101 \cdot (f(x))^{100} \cdot f'(x) = f(x)$

$\int 101 (f(x))^{99} \cdot f'(x) \cdot dx = \int 1 \cdot dx$

$\frac{101\, (f(x))^{100}}{100} = x + c$

$f(0) = 1$
$\Rightarrow c = 1$

$\frac{f(10)^{100}}{100} = \frac{x}{101} + c$

$f(101)^{100} = \boxed{100 + c = 100 + 1 = 101}$

$$f(x) = \sin x + \int_{-\pi/2}^{\pi/2} \sin x \cdot dt + \int_{-\pi/2}^{\pi/2} t\, f(t) \cdot dt$$

$$= \sin x + (\sin x)\pi + \int_{-\pi/2}^{\pi/2} \underset{\downarrow \atop \text{I}}{t}\; \underset{\downarrow \atop \text{II}}{f(t)} \cdot dt$$

$$f(x) = \sin x\,(1+\pi) + A$$

$$A = \int_{-\pi/2}^{\pi/2} t \cdot f(t) \cdot dt$$

$$A = \int_{-\pi/2}^{\pi/2} t \cdot (\sin t\,(1+\pi) + A) \cdot dt$$

$$A = \int_{-\pi/2}^{\pi/2} t \sin t\,(1+\pi) \cdot dt + \int_{-\pi/2}^{\pi/2} At \cdot dt$$

$$A = (1+\pi) \int_{-\pi/2}^{\pi/2} t \sin t\, dt$$

$$A = 2(1+\pi) \int_{0}^{\pi/2} t \sin t \cdot dt$$

$$A = 2\,(1+\pi)\,(\sin t - t\cos t)\Big|_0^{\pi/2}$$

$$A = 2\,(1+\pi)$$

$$f(x) = (1+\pi)\sin x + 2\,(1+\pi)$$

$$f(x) = (1+\pi)(\sin x + 2)$$

$$\boxed{f(x)\Big|_{max} = 3(1+\pi)\ ,\quad f(x)\Big|_{min} = (1+\pi)}$$

FORM-II $\int_a^b f(x,t)\cdot dx = I(t)$ where a and b limits of x and t is a parameter independent of x,

Then $$\frac{d[I(t)]}{dt} = \int_a^b f'(x,t)\,dx.$$

Here $f'(x,t)$ = diff of $f(x,t)$ w.r.t. 't' considering x as a constant.

Question

$$\int_{\pi/6}^{\pi/3} \sin(tx)\cdot dx = I(x,t)$$

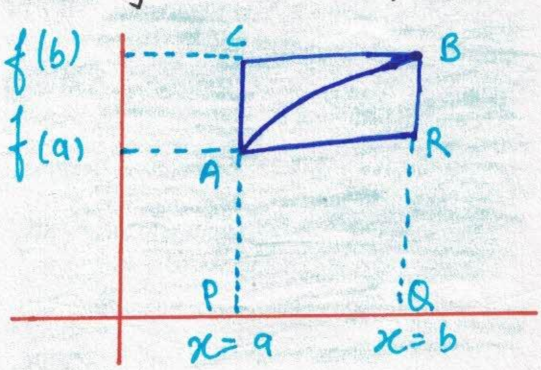
$$\frac{d[I(x,t)]}{dt} = \int_{\pi/6}^{\pi/3} x\cos(tx)\cdot dx$$

$$= \left[\frac{x\sin tx}{t} - \frac{1}{t}\int \sin tx\cdot dx\right]_{\pi/6}^{\pi/3}$$

$$= \left[\frac{x\sin tx}{t} + \frac{\cos tx}{t^2}\right]_{\pi/6}^{\pi/3}$$

ESTIMATION PROPERTIES

CASE-I If $f(x)$ is monotonically increasing fn in $[a,b]$, then-

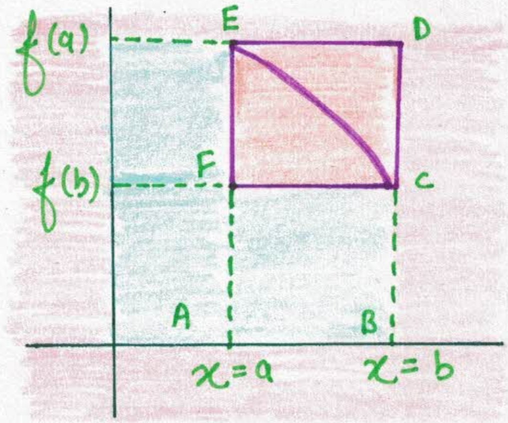


$$(b-a)f(a) < \int_a^b f(x)\cdot dx < (b-a)f(b)$$

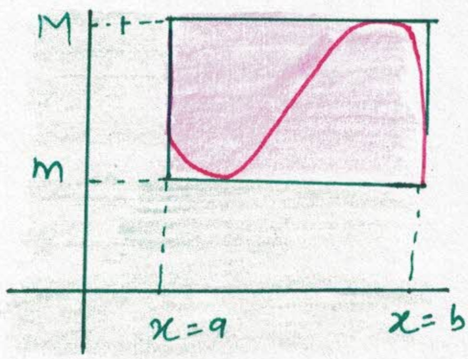
CASE-II If $f(x)$ is monotonically decreasing fn in $[a, b]$

Then -

$$(b-a)f(b) < \int_a^b f(x)\,dx < (b-a)f(a)$$



CASE-III



$$(b-a)m < \int_a^b f(x)\cdot dx < (b-a)M$$

$\downarrow$ min of $f(x)$ in $[a, b)$

$\downarrow$ max. of $f(x)$ in $(a, b]$

Question Prove that $\int_{\pi/4}^{\pi/2} (\sin x)^{10}\cdot dx$ is b/w $\pi/128$ to $\pi/4$.

$$f(\pi/4) = \left(\frac{1}{\sqrt{2}}\right)^{10} = \left(\frac{1}{2}\right)^5 = \frac{1}{16\times 2}$$

$a - b = \pi/4$

$f(b) = 1$

$$f(a) = \left(\frac{\pi}{4}\right)\left(\frac{1}{16}\right)\times\frac{1}{2} = \frac{\pi}{128}$$

$$\frac{\pi}{128} < f(x) < \pi/4$$

$$\int_0^{\pi/2} \frac{\sin x}{x}\cdot dx \qquad [-1, 1] \quad (0, \pi] \qquad \frac{x\cos x - \sin x}{x^2} \text{ Decreasing fn}$$

$$f(\pi/2) = \frac{2}{\pi} \qquad f(0) = 1$$

$$\frac{2}{\pi}\cdot\frac{\pi}{2} < f(x) < 1\cdot\pi/2$$

$$\boxed{1 < f(x) < \pi/2}$$

WALLI'S FORMULA...

m and n are integers.

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x \cdot dx = \left[\frac{((n-1)(n-3)(n-5)\ldots\ldots 1 \text{ or } 2)((m-1)(m-3)\ldots\ldots 1 \text{ or } 2)}{\{(m+n)(m+n-2)(m+n-4)\ldots\ldots\ldots 1 \text{ or } 2}\right] \times K$$

$$K = \begin{cases} \pi/2 & \text{if } m, n \text{ both even} \\ 1 & \text{otherwise} \end{cases}$$

$$\int_0^{\pi/2} \sin^2 x \cos x \cdot dx = \frac{1 \cdot 1}{3 \cdot 1} \cdot 1 = \frac{1}{3}$$

$$\int_0^{\pi/2} \sin^7 x \cos x \cdot dx = \frac{(6)\cancel{(4)}(2)}{8 \cdot 6 \cdot \cancel{4} \cdot 2} \cdot 1 = \frac{1}{8}$$

$$\int_0^{\pi/2} \sin^4 x \cos^2 x \cdot dx = \frac{\cancel{(3)}(1) \cdot 1}{\underset{2}{\cancel{(6)}}(4)(2)} \cdot \frac{\pi}{2}$$

$$\boxed{= \frac{\pi}{32}}$$